

SYMMETRIC POLYNOMIALS AND  $U_q(\widehat{sl}_2)$ 

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ABSTRACT. We study the explicit formula of Lusztig's integral forms of the level one quantum affine algebra  $U_q(\widehat{sl}_2)$  in the endomorphism ring of symmetric functions in infinitely many variables tensored with the group algebra of  $\mathbb{Z}$ . Schur functions are realized as certain orthonormal basis vectors in the vertex representation associated to the standard Heisenberg algebra. In this picture the Littlewood-Richardson rule is expressed by integral formulas, and is used to define the action of Lusztig's  $\mathbb{Z}[q, q]$ -form of  $U_q(\widehat{sl}_2)$  on Schur polynomials.

## 1. INTRODUCTION

The relation between vertex representations and the symmetric functions is one of the interesting aspects of affine Kac-Moody algebras, quantum affine algebras and the vertex operator algebra. In the early 1980's the Kyoto school [DJKM] found that the polynomial solutions of KP hierarchies are obtained by Schur polynomials. This breakthrough was achieved in their work of describing the KP and KdV hierarchies in terms of affine Lie algebras. On the other hand I. Frenkel [F1] identified the two constructions of the affine Lie algebras via the vertex operators, which put the boson-fermion correspondence in a rigorous formulation. Later I. Frenkel further used this idea [F2] to find out that the boson-fermion correspondence can also be used to realize the Schur polynomials in the study of irreducible characters of the symmetric group  $S_n$ . Schur functions also appeared in Lepowsky and Primc construction [LP] of certain bases for higher level representations of the affine Lie algebra  $\widehat{sl}(2)$ .

In [J1, J2] the author advanced the vertex operator approach to the symmetric functions and realized Schur's Q-functions, which assume the same role of Schur functions in the context of (non-trivial) irreducible characters of the double covering groups of  $S_n$ , and more generally Hall-Littlewood polynomials were treated in the same way. These families of symmetric polynomials appear naturally as orthogonal bases in the vertex representation. In [J1] I. Frenkel's original work on  $S_n$  was reviewed together with Bernstein's formula (see [Z]). The exact formulation of Hall-Littlewood polynomials in terms of the boson-fermion correspondence appeared later in the author's work [J5]. Since Macdonald's seminal extension [M] of Hall-Littlewood polynomials it immediately brought up the question of the vertex operator approach to the more general Macdonald polynomials.

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In [J4] we also found that the vertex operator realization of certain Macdonald polynomials is governed by the basic hypergeometric functions of type  ${}_4\phi_3$ . It was realized [J3] that the complete vertex operator realization of Macdonald polynomials is related to several new questions, among them the most interesting question was how one could realize the Macdonald polynomials in the vertex representations [FJ] of the quantum affine algebra  $U_q(\widehat{sl}_2)$ . In this case the underlying symmetry of the analogous family of symmetric functions are expressed by certain infinite dimensional quadratic algebras generalizing the Clifford-Weyl algebra.

Recently J. Beck, I. Frenkel and the author [BFJ] have used vertex operators to study the canonical bases for the level one irreducible modules for the quantum affine algebra  $U_q(\widehat{sl}_2)$ . Macdonald polynomials (zonal case) are seen as some "canonical" bases sitting between Kashiwara and Lusztig's canonical and dual canonical bases [L, K]. This essentially answered the question about the vertex realization of Macdonald polynomials. The Macdonald basis constructed in [BFJ] also satisfy the characteristic properties of bar invariance and orthogonality under the Kashiwara form. The transition matrix from the canonical basis to Macdonald basis is triangular, integral and bar-invariant. We also conjectured its positivity. Since the transition matrix from Macdonald polynomials to (modified) Schur polynomials is also triangular, it is natural to ask how does the quantum affine algebra act on Schur polynomials.

Our first goal in this paper is to answer the question of realizing the quantum affine algebra  $U_q(\widehat{sl}_2)$  by Schur functions. We give explicit formulas to realize the level one representation with the help of Littlewood-Richardson rule. In order to do this we first redevelop the vertex approach to Schur functions. Mixed products of vertex operators and dual vertex operators are expressed in terms of Schur functions. We showed that the underlying symmetry of the Young tableaux of the various classical symmetric functions are governed by certain Clifford-Weyl type algebras, where the simplest case of  $S_n$ -symmetry or the linkage symmetry corresponds to the infinite dimensional Clifford algebra.

Using this the quantum affine algebra  $U_q(\widehat{sl}_2)$  is represented as  $\mathbb{Z}[q, q^{-1}]$ -linear operators on the ring of symmetric functions. Moreover, the lattice of the irreducible module  $V(\Lambda_i)$  ( $i = 0, 1$ ):

$$\mathcal{L}(\Lambda_i) = \Lambda(x_1, \dots, x_n \dots) \otimes \mathbb{C}[\mathbb{Z}\alpha]e^{i\alpha/2}$$

is shown to be an integral lattice for Lusztig's  $\mathbb{Z}[q, q^{-1}]$ -form of  $U_q(\widehat{sl}_2)$  (generated by divided powers). On the other hand Beck, Chari and Pressley [BCP] have shown that a related lattice  $\mathcal{L}(\Lambda_i)' = \Lambda(x'_1, \dots, x'_n \dots) \otimes \mathbb{C}[\mathbb{Z}\alpha]e^{i\alpha/2}$  is contained in  $V(\Lambda_i)_{\mathbb{Z}[q, q^{-1}]}$ , where  $\sum_i x_i'^n = (1 - q^{4n}) \sum_i x_i^n$ . The explicit realization of the Lusztig's integral lattice  $V(\Lambda_i)_{\mathbb{Z}[q, q^{-1}]}$  will be treated elsewhere.

We further write down the action of divided powers of current operators in terms of Littlewood-Richardson rule. Our method stems from the trick [J3] of expressing Schur functions in terms of a deformed Heisenberg algebra inside  $U_q(\widehat{sl}_2)$ . Thus we are able to write down everything explicitly using the vertex representation, which partly generalizes Garland's work [G] (see also [CP]). The method in this paper can also be generalized to quantum affine algebras of ADE types, and this may provide more information about Schur functions as crystal bases [BCP]. Similarly our formulas might help to understand the positivity conjecture [BFJ].

Our second goal is to give a vertex operator approach to the Littlewood-Richardson rule. In particular an integral formula for the Littlewood-Richardson rule is found. In the fermionic picture of the crystal basis of quantum affine algebras Misra and Miwa [MM] used some insertion and deletion procedure on Young tableaux to describe the action of quantum affine algebra at  $q = 0$ . The explicit rule in the fermionic construction [H] suggests that there should be a corresponding rule in the homogeneous construction. This turns out to be an explicit formula in terms of the Littlewood-Richardson rule in our work (see Proposition 2.2 and Theorem 3.2).

Our explicit formulas of the divided powers of Chevalley generators suggest that there should be corresponding formulas in the fermionic picture. The answer to this question will be helpful to understand the real meaning of Littlewood-Richardson rule in the boson-fermion correspondence.

The paper is organized as follows. In section 2 we redevelop our vertex operator approach to Schur polynomials and express all mixed products of dual vertex operators in terms of Schur functions ( $q = 1$  case). In section 3 we first recall the Frenkel-Jing vertex representation of  $U_q(\widehat{sl}_2)$  and obtain explicit formulas for the Drinfeld generators in terms of the Schur basis constructed in Section 2. We then enter the Littlewood-Richardson rule to give the formulas for the divided powers of the current operators  $X_n^\pm$ . In the last section (Sec. 4) we recast the action in terms of the Chevalley generators, which provides a simple combinatorial model for the homogeneous picture of the basic module for  $U_q(\widehat{sl}_2)$ . We show that the lattice of Schur bases contains Lusztig's integral lattice of divided powers.

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## 2. SCHUR FUNCTIONS AND VERTEX OPERATORS

Let  $\Lambda_F$  be the ring of symmetric functions in infinitely many variables  $x_1, x_2, \dots$  over the field  $F$ . In this section we take  $F = \mathbb{Q}$ , and later we will take  $F = \mathbb{Q}(q^{1/2}, q^{-1/2})$ .

A partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$  of  $n$ , denoted  $\lambda \vdash n$ , is a special decomposition of  $n$ :  $n = \lambda_1 + \dots + \lambda_l$  with  $\lambda_1 \geq \dots \geq \lambda_l \geq 1$ .  $l$  is called the length of  $\lambda$ . We will identify  $(\lambda_1, \dots, \lambda_l)$  with  $(\lambda_1, \dots, \lambda_l, 0, \dots, 0)$  if we want to view  $\lambda$  in  $\mathbb{Z}^n$  when  $n \geq l(\lambda)$ . Sometime we prefer to use another notation for  $\lambda$ :  $(1^{m_1} 2^{m_2} \dots)$  where  $m_i$  is the number of times that  $i$  appears among the parts of  $\lambda$ . The set of partitions will be denoted by  $\mathcal{P}$ .

There are several well-known bases in  $\Lambda_F$  parameterized by partitions: the power sum symmetric functions  $p_\lambda = p_{\lambda_1} \dots p_{\lambda_l}$  with  $p_n = \sum x_i^n$  ( $\mathbb{Q}$ -basis), the monomial symmetric functions  $m_\lambda(x_1, \dots, x_n) = \sum_{\sigma} x_1^{\sigma(\lambda_1)} \dots x_n^{\sigma(\lambda_n)}$ , where  $\sigma$  runs through distinct permutations of  $\lambda$  as tuples; and the Schur functions  $s_\lambda$  which are over  $\mathbb{Z}$ . In terms of finitely many variables Schur function is given by the Weyl character formula:

$$(2.1) \quad s_\lambda(x_1, \dots, x_n) = \frac{\sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) x^{\sigma(\lambda+\delta)}}{\prod_{i < j} (x_i - x_j)},$$

where  $\delta = (n-1, n-2, \dots, 1, 0)$ ,  $\lambda = (\lambda_1, \dots, \lambda_n)$ . Here some  $\lambda_i$  may be zero.

Note that this formula also works for any tuple  $\mu$ . In general  $s_\mu = 0$  or  $(-1)^{l(\sigma)} s_\lambda$ , where  $\lambda = \sigma(\mu + \delta) - \delta$  for some permutation  $\sigma$  and  $l(\sigma)$  is the length of the permutation  $\sigma$ . This important property is still true for the Schur function in infinitely many variables, though there is a less satisfactory formula in algebraic combinatorics in that case. We will see that this symmetry property is manifested in our vertex operator approach.

Let's recall the vertex operator approach to Schur functions [J1]. Let  $\{b_n | n \neq 0\} \cup \{c\}$  be the set of generators of the Heisenberg algebra with defining relations

$$(2.2) \quad [b_m, b_n] = m\delta_{m,-n}c, \quad [c, b_m] = 0.$$

The Heisenberg algebra has a canonical natural representation in the  $\mathbb{Q}$ -space  $V = \text{Sym}(b_{-n}'s)$ , the symmetric algebra generated by the  $b_{-n}$ ,  $n \in \mathbb{N}$ . The action is given by

$$(2.3) \quad b_{-n}.v = b_{-n}v, \quad b_n.v = n \frac{\partial v}{\partial b_{-n}},$$

$$(2.4) \quad c.v = v.$$

It is clear that 1 is the highest weight vector in  $V$ .

Let us introduce two vertex operators (cf.  $t = 0$  in [J2]):

$$(2.5) \quad \begin{aligned} S(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{b_{-n}}{n} z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{b_n}{n} z^{-n}\right) \\ &= \sum_{n \in \mathbb{Z}} S_n z^{-n}, \end{aligned}$$

$$(2.6) \quad \begin{aligned} S^*(z) &= \exp\left(-\sum_{n=1}^{\infty} \frac{b_{-n}}{n} z^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{b_n}{n} z^n\right) \\ &= \sum_{n \in \mathbb{Z}} S_n^* z^n. \end{aligned}$$

It follows that for  $n \geq 0$

$$S_n.1 = \delta_{n,0}, \quad S_{-n}^*.1 = \delta_{n,0}.$$

There is a natural hermitian inner product on  $V$  given by

$$(2.7) \quad b_n^* = b_{-n}.$$

The elements  $b_{-\lambda} = b_{-\lambda_1} \cdots b_{-\lambda_l} (\lambda \vdash n)$  span an orthogonal basis in  $V$  and

$$(2.8) \quad (b_{-\lambda}, b_{-\mu}) = \delta_{m,n} z_\lambda,$$

where  $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$  for  $\lambda = (1^{m_1} 2^{m_2} \dots)$ .

The vertex space  $V$  is isomorphic to the ring  $\Lambda_{\mathbb{Q}}$  under the characteristic map:

$$b_{-\lambda} = b_{-\lambda_1} \cdots b_{-\lambda_l} \longrightarrow p_\lambda.$$

The characteristic map turns out to be also isometric (see [J1]).

**Definition 2.1.** The polynomial  $s_n = S_{-n}.1$  is called the  $n^{th}$  complete homogeneous symmetric polynomial in the  $b_{-n}$ .

*Remark 2.2.* We do not attempt to distinguish the term "symmetric function" from that of "symmetric polynomial". It is apparant that the symmetry is only with respect to the variables  $x_1, x_2, \dots$  and not with respect to the power sum variable.

The generating function of  $s_n$  is given by:

$$(2.9) \quad \sum_{n=0}^{\infty} s_n z^n = \exp\left(\sum_{n=1}^{\infty} \frac{b_{-n}}{n} z^n\right).$$

**Lemma 2.3.** [J2] *The components of  $S(z)$  and  $S^*(z)$  satisfy the following commutation relations.*

$$\begin{aligned} S_m S_n + S_{n-1} S_{m+1} &= 0, & S_m^* S_n^* + S_{n+1}^* S_{m-1}^* &= 0, \\ S_m S_n^* + S_{n-1}^* S_{m-1} &= \delta_{m,n}. \end{aligned}$$

Two  $l$ -tuple  $\mu$  and  $\lambda$  are related if there is a permutation  $\sigma$  such that  $\mu + \delta = \sigma(\lambda + \delta)$ . If  $\mu_i = \mu_{i+1} - 1$ , then  $\mu + \delta = \sigma(\mu + \delta)$  for  $\sigma = (i, i+1)$ . If there exists an old permutation  $\sigma$  such that  $\mu + \delta = \sigma(\mu + \delta)$  then we say that  $\mu$  is degenerate. For any non-degenerate  $l$ -tuple  $\mu = (\mu_1, \dots, \mu_l)$  in the general position such that no two parts  $\mu_i, \mu_j$  ( $i < j$ ) satisfying  $\mu_i = \mu_j - (j - i)$  then there exists uniquely a partition  $\lambda$  such that  $\mu + \delta = \sigma(\lambda + \delta)$ . We remark that the degeneracy condition corresponds to whether a weight belongs to some wall of the Weyl chambers for  $\mathfrak{S}_l$ . In the latter context the symmetry is called the linkage symmetry. For simplicity we let  $\text{sgn}(\mu) = 0$  if the  $l$ -tuple  $\mu$  is degenerate.

**Theorem 2.4.** [J1, J5] *Under the mapping  $b_{-n} \rightarrow p_n$ , the space  $V$  is isometrically isomorphic to  $\Lambda_{\mathbb{Q}}$ . The set  $\{h_{-\lambda} = s_{\lambda_1} s_{\lambda_2} \dots s_{\lambda_l} : \lambda \vdash n, n \in \mathbb{Z}_+\}$  and  $\{S_{-\lambda_1} S_{-\lambda_2} \dots S_{-\lambda_l} \cdot 1 : \lambda \vdash n, n \in \mathbb{Z}_+\}$  both form  $\mathbb{C}$ -linear bases. Moreover the basis  $\{S_{-\lambda_1} S_{-\lambda_2} \dots S_{-\lambda_l} \cdot 1\}$  is orthonormal and expressed explicitly by:*

$$(2.10) \quad S_{-\lambda} \cdot 1 := S_{-\lambda_1} S_{-\lambda_2} \dots S_{-\lambda_l} \cdot 1 = \det(s_{\lambda_i - i + j}) = s_{\lambda},$$

where  $s_{\lambda}$  is the Schur function in the  $b_{-n}$  under the isomorphism between  $\Lambda_{\mathbb{Q}}$  and  $V$ .

*Proof.* To show the  $\mathfrak{S}_n$ -symmetry we consider the modified vertex operators associated with the root lattice  $\mathbb{Z}\alpha$  with  $(\alpha|\alpha) = 1$ . Let  $\tilde{V} = V \otimes \mathbb{C}[\mathbb{Z}\alpha]$ , where  $\mathbb{C}[\mathbb{Z}\alpha]$  is the group algebra generated by  $e^{m\alpha}, m \in \mathbb{Z}$ . Define

$$(2.11) \quad \overline{S}(z) = S(z) e^{\alpha} z^{\partial} = \sum_{n \in \mathbb{Z} + 1/2} \overline{S}_n z^{-n-1/2},$$

$$(2.12) \quad \overline{S}^*(z) = S^*(z) e^{-\alpha} z^{-\partial} = \sum_{n \in \mathbb{Z} + 1/2} \overline{S}_n^* z^{n-1/2},$$

where the operators  $e^{\alpha}$  and  $z^{\partial}$  act on  $\mathbb{C}[\mathbb{Z}\alpha]$  as follows:

$$e^{\alpha} e^{m\alpha} = e^{(m+1)\alpha}, \quad z^{\partial} e^{m\alpha} = z^m e^{m\alpha},$$

The components satisfy the Clifford algebra relations

$$(2.13) \quad \{\overline{S}_m, \overline{S}_n\} = \{\overline{S}_m^*, \overline{S}_n^*\} = 0,$$

$$(2.14) \quad \{\overline{S}_m, \overline{S}_n^*\} = \delta_{m,n},$$

where  $m, n \in \mathbb{Z} + 1/2$ .

For any degenerate  $l$ -tuple  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$  satisfying  $\lambda_i = \lambda_j - j + i$  for some  $i$  and  $j$  we have

$$\begin{aligned} S_{-\lambda_1} \cdots S_{-\lambda_l} \cdot 1 &= \overline{S}_{-\lambda_1+1/2} \overline{S}_{-\lambda_2-1/2} \cdots \overline{S} \overline{F}_{-\lambda_l-l+1/2} e^{-l\alpha} \\ &= \overline{S}_{-\lambda+\delta-(m-1/2)\mathbf{1}} \\ &= 0 \end{aligned}$$

due to  $\lambda + \delta = (i, j)(\lambda + \delta)$  and (2.13-2.14). Here  $\mathbf{1}$  denotes the integer vector  $(1, 1, \dots, 1)$ , and similarly  $m\mathbf{1} = (m, m, \dots, m)$ . In general for any non-degenerate  $l$ -tuple we have

$$(2.15) \quad S_{-\mu_1} \cdots S_{-\mu_l} \cdot 1 = \text{sgn}(\mu) S_{-\lambda_1} \cdots S_{-\lambda_l} \cdot 1$$

where  $\lambda$  is the partition related to  $\mu$ :  $\lambda = \sigma(\mu + \delta) - \delta$ , and the sign  $\text{sign}(\mu) = (-1)^{l(\sigma)}$ . See [J1] for details.  $\square$

The following result will be useful in our discussion.

**Proposition 2.1.** *Let  $\delta = (n-1, n-2, \dots, 1, 0)$ . The operator products  $S(z_1)S(z_2) \cdots S(z_n)z^\delta$  and  $S^*(z_1)S^*(z_2) \cdots S^*(z_n)z^\delta$  are skew-symmetric under the action of  $\mathfrak{S}_n$ . For any  $w \in \mathfrak{S}_n$  we have*

$$\begin{aligned} S(z_{w(1)})S(z_{w(2)}) \cdots S(z_{w(n)})z^{w(\delta)} &= (-1)^{l(w)} S(z_1)S(z_2) \cdots S(z_n)z^\delta \\ S^*(z_{w(1)})S^*(z_{w(2)}) \cdots S^*(z_{w(n)})z^{w(\delta)} &= (-1)^{l(w)} S^*(z_1)S^*(z_2) \cdots S^*(z_n)z^\delta \end{aligned}$$

**Definition 2.5.** For an  $l$ -tuple  $\mu$  we define the Schur function  $s_\mu$  to be the symmetric function corresponding to  $S_{-\mu_1} \cdots S_{-\mu_l} \cdot 1$  under the characteristic map. We will simply write

$$s_\mu = S_{-\mu} \cdot 1 = S_{-\mu_1} \cdots S_{-\mu_l} \cdot 1.$$

The following fact follows easily from Theorem 2.4 .

$$(2.16) \quad s_\mu = \begin{cases} 0 & \mu \text{ is degenerate} \\ \text{sgn}(\sigma) s_\lambda & \lambda = \sigma(\mu + \delta) - \delta, \lambda \in \mathcal{P} \end{cases}.$$

For convenience we denote the partition  $\lambda$  associated to the tuple  $\mu$  by  $\lambda = \pi(\mu)$ .

We remark that the characteristic map can be defined over  $\mathbb{Z}$  if we do it on the basis of homogeneous polynomials or Schur polynomials. Correspondingly we have similar results for the dual vertex operators  $S^*(z)$ . Here the vector  $S_n^* \cdot 1$  corresponds to the elementary symmetric function  $(-1)^n e_n = (-1)^n s_{(1^n)}$ .

For a partition  $\lambda = (\lambda_1, \dots, \lambda_l)$  we denote by  $\lambda' = (\lambda'_1, \dots, \lambda'_k)$  the dual partition, where  $\lambda'_i = \text{Card}\{j : \lambda_j \geq i\}$ . We also denote by  $(\lambda, \mu)$  the juxtaposition of two partitions or tuples. Note that  $(\lambda, \mu)$  is generally not a partition.

**Theorem 2.6.** *The set  $\{S_{\lambda_1}^* \cdots S_{\lambda_l}^* \cdot 1 \mid \lambda \vdash n, n \in \mathbb{Z}_+\}$  also forms an orthonormal basis.*

$$(2.17) \quad S_{\lambda_1}^* S_{\lambda_2}^* \cdots S_{\lambda_l}^* \cdot 1 = (-1)^{|\lambda|} s_{\lambda'} = (-1)^{|\lambda|} \det(s_{\lambda'_i - i + j}),$$

where  $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ . Moreover we have

$$\begin{aligned} S_{-\mu_1} \cdots S_{-\mu_l} S_{\nu_1}^* \cdots S_{\nu_k}^* \cdot 1 &= (-1)^{|\nu|} \text{sgn}(\lambda, \mu) s_{\pi(\mu, \nu')} \\ S_{\mu_1}^* \cdots S_{\mu_k}^* S_{-\nu_1} \cdots S_{-\nu_l} \cdot 1 &= (-1)^{|\mu|} \text{sgn}(\mu, \nu) s_{\pi(\mu, \nu')} \end{aligned}$$

where  $(\mu, \nu)$  is the juxtaposition of  $\mu$  and  $\nu$ , and the associated partition  $\pi(\mu, \nu)$  is obtained by  $\pi(\mu, \nu) = \sigma((\mu, \nu) + \delta) - \delta$  for some  $\sigma \in \mathfrak{S}_{l+k}$ .

As a consequence of Theorem 2.6 it follows that  $S_{-\lambda}.1 = (-1)^{|\lambda|} S_{\lambda}^*.1$ . In summary we have the following symmetry property on the parts.

$$(2.18) \quad s_{\mu'} = \begin{cases} 0 & \mu \text{ is degenerate} \\ sgn(\sigma) s_{\pi(\mu)'} & \pi(\mu) = \sigma(\mu + \delta) - \delta, \pi(\mu) \in \mathcal{P} \end{cases}.$$

Note that we apply the action of  $\mathfrak{S}_n$  first on the  $n$ -tuples and then follow by the duality.

**Example 2.7.**

$$\begin{aligned} S_{-1} S_2^* S_2^*.1 &= S_{-1} S_{-2} S_{-2}.1 = 0, \\ S_{-1} S_2^* S_1^* S_1^* S_1^*.1 &= -S_{-1} S_{-4} S_{-1}.1 = S_{-3} S_{-2} S_{-1}.1 = s_{(3,2,1)}, \end{aligned}$$

where  $(1, 4, 1) + \delta = (3, 5, 1) \sim (5, 3, 1) = (3, 2, 1) + \delta$ .

To close this section we derive the Littlewood-Richardson rule in our picture, which will be used later to realize the action of  $U_q(\widehat{sl}_2)$  on  $\Lambda_{\mathbb{Q}[q, q^{-1}]}$ .

**Proposition 2.2.** *Let  $\lambda$  and  $\mu$  be two partitions of lengths  $m$  and  $n$  respectively. Then  $s_{\lambda} s_{\mu} = \sum C_{MN} s_{(\lambda+M, \mu-N)}$  where  $C_{MN}$  is the number of integral matrices  $(k_{ij})$  such that*

$$\begin{aligned} (k_{11} + k_{12} + \cdots + k_{1n}, \dots, k_{m1} + \cdots + k_{mn}) &= M, \\ (k_{11} + k_{21} + \cdots + k_{m1}, \dots, k_{1n} + k_{2n} + \cdots + k_{mn}) &= N. \end{aligned}$$

*In particular the Schur functions form a  $\mathbb{Z}$ -basis in  $\Lambda$ .*

*Proof.* It follows from definition that

$$s_{\lambda} s_{\mu} = \int S(z_1) \cdots S(z_m).1 S(w_1) \cdots S(w_n).1 z^{-\lambda} w^{-\mu} \frac{dz}{z} \frac{dw}{w}.$$

Observe that

$$\begin{aligned} &S(z_1) \cdots S(z_m).1 S(w_1) \cdots S(w_n).1 \\ &= \prod_{i,j} \left(1 - \frac{w_j}{z_i}\right)^{-1} S(z_1) \cdots S(z_m) S(w_1) \cdots S(w_n). \end{aligned}$$

As an infinite series in  $|w_j| < |z_i|$  we have

$$\begin{aligned} \prod_{i,j} \left(1 - \frac{w_j}{z_i}\right)^{-1} &= \prod_{i,j} \left(1 + \frac{w_j}{z_i} + \left(\frac{w_j}{z_i}\right)^2 + \cdots\right) \\ &= \prod_{i,j} \left(\sum_{k_{ij}} w_j^{k_{ij}} z_i^{-k_{ij}}\right) \\ &= \prod_{k=(k_{ij})} w_1^{k_{.1}} \cdots w_n^{k_{.n}} z_1^{-k_{1.}} \cdots z_m^{-k_{m.}}, \end{aligned}$$

with  $k_{.j} = k_{1j} + \cdots + k_{mj}$ ,  $k_{i.} = k_{i1} + \cdots + k_{in}$ ,  $k_{ij} \geq 0$ .

Plugging the expansion into the integral and invoking Theorem (2.4). We prove the proposition.  $\square$

*Remark 2.8.* The number  $C_{MN}$  is equal to the index of  $\mathfrak{S}_{\lambda} \pi \mathfrak{S}_{\mu}$  in  $\mathfrak{S}_n$  [JK]. We can also write:

$$s_{\lambda} s_{\mu} = \prod_{i,j} (1 - R_{ij})^{-1} s_{(\lambda, \mu)},$$

where  $R_{ij}$  is the raising operator defined on the parts of the Schur functions:  
 $R_{ij}s(\dots, \lambda_i, \dots, \lambda_j, \dots) = s(\dots, \lambda_i+1, \dots, \lambda_j-1, \dots)$ .

### 3. QUANTUM AFFINE ALGEBRA $U_q(\widehat{sl}_2)$ .

For  $n \in \mathbb{Z}_+$  we define  $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ . The  $q$ -factorial  $[n]!$  denotes  $[n][n-1] \cdots [2][1]$  and then the  $q$ -Gaussian numbers are defined naturally by  $\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}$  for  $n \geq m \geq 0$ . By convention  $[0] = [1] = 1$ . For an element  $a$  in an algebra over  $\mathbb{Z}[q, q^{-1}]$  we use  $a^{(n)}$  to denote the divided power  $\frac{a^n}{[n]!}$ .

The quantum affine algebra  $U_q(\widehat{sl}_2)$  is the associative algebra generated by Chevalley generators  $e_i, f_i, K_i$  ( $i = 0, 1$ ) and  $q^d$  subject to the following defining relations.

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1} K_i = 1, \quad q^d q^{-d} = q^{-d} q^d = 1, \\ K_i K_j &= K_j K_i, \quad q^d K_i^{\pm 1} = K_i^{\pm 1} q^d, \\ K_i e_j K_i^{-1} &= q^{a_{ij}} e_j, \quad K_i f_j K_i^{-1} = q^{-a_{ij}} f_j, \\ q^d e_i q^{-d} &= q^{\delta_{i,0}} e_i, \quad q^d f_i q^{-d} = q^{-\delta_{i,0}} f_i, \\ [e_i, f_j] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r e_i^{(r)} e_j e_i^{(1-a_{ij}-r)} &= 0 \quad \text{if } i \neq j, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r f_i^{(r)} f_j f_i^{(1-a_{ij}-r)} &= 0 \quad \text{if } i \neq j. \end{aligned}$$

where  $(a_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$  is the extended Cartan matrix [Ka]. The central element  $K_0 K_1 = q^c$  acts as an integral power of  $q$  on an integrable module of  $U_q(\widehat{sl}_2)$ .

We will be interested in level one ( $c = 1$ ) irreducible representations, which had been realized by vertex operators in [FJ]. For our purpose we will recall the vertex representations in a rescaled form.

Let  $\{a_n | n \neq 0\} \cup \{c\}$  be the generators the Heisenberg algebra  $U_q(\widehat{sl}_2)$  with the defining relation.

$$(3.1) \quad [a_m, a_n] = \delta_{m,-n} \frac{m}{1 + q^{2|m|c}}.$$

Following Frenkel-Jing [FJ] the level one irreducible representation  $V(\Lambda_i)$  is realized on the vertex representation space  $Sym(a_{-n}'s) \otimes \mathbb{C}[\mathbb{Z}\alpha]e^{i\alpha/2}$  ( $i = 0, 1$ ) at  $c = 1$ . Here  $Sym(a_{-n}'s)$  denotes the symmetric algebra generated by the Heisenberg generators  $a_{-n}$ 's. The element  $e^{i\alpha/2}$  (the highest weight vector) will be formally adjoined to  $\mathbb{C}[\mathbb{Z}\alpha]$ .

We define two kinds of operators on the vector space  $\mathbb{C}[\mathbb{Z}\alpha]e^{i\alpha/2} = \langle e^{m\alpha} e^{i\alpha/2} | m \in \mathbb{Z} \rangle$ :

$$(3.2) \quad e^{n\alpha} \cdot e^{m\alpha} e^{i\alpha/2} = e^{(m+n)\alpha} e^{i\alpha/2}, \quad n \in \mathbb{Z},$$

$$(3.3) \quad \partial \cdot e^{m\alpha} e^{i\alpha/2} = (2m + i) e^{m\alpha} e^{i\alpha/2}.$$



In particular  $\mathbb{C}[\mathbb{Z}\alpha]e^{i\alpha/2}$  is a  $\mathbb{C}[\mathbb{Z}\alpha]$ -module.

We define the vertex operators associated to  $U_q(\widehat{sl}_2)$  by:

$$(3.4) \quad X^+(z) = \exp\left(\sum_{n=1}^{\infty} \frac{(1+q^{2n})q^{-n}}{n} a_{-n} z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{(1+q^{2n})q^{-n}}{n} a_n z^{-n}\right) e^{\alpha} z^{\partial}$$

$$= \sum_{n \in \mathbb{Z}} X_n^+ z^{-n-1}$$

$$(3.5) \quad X^-(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{1+q^{2n}}{n} a_{-n} z^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{1+q^{2n}}{n} a_n z^{-n}\right) e^{-\alpha} z^{-\partial}$$

$$= \sum_{n \in \mathbb{Z}} X_n^- z^{-n-1}$$

The normal order of vertex operator products is defined by rearranging the exponential factors. For example,

$$\begin{aligned} & : X^+(z) X^+(w) : \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{(1+q^{2n})q^{-n}}{n} a_{-n} (z^n + w^n)\right) \exp\left(-\sum_{n=1}^{\infty} \frac{(1+q^{2n})q^{-n}}{n} a_n (z^{-n} + w^{-n})\right) \\ & \quad e^{2\alpha} z^{\partial} w^{\partial}. \end{aligned}$$

The components of the Drinfeld generators satisfy some quadratic relations (Serre relations).

**Lemma 3.1.** [FJ] *The components of  $X^{\pm}(z)$  satisfy the following commutation relations.*

$$\begin{aligned} X_m^{\pm} X_n^{\pm} - q^{\pm 2} X_n^{\pm} X_m^{\pm} &= q^{\pm 2} X_{m-1}^{\pm} X_{n+1}^{\pm} - X_{n+1}^{\pm} X_{m-1}^{\pm}, \\ X_m^+ X_n^- - X_n^- X_m^+ &= \frac{1}{q - q^{-1}} \left( \psi_{m+n} q^{(m-n)/2} - \phi_{m+n} q^{-(m-n)/2} \right), \end{aligned}$$

where the polynomials  $\psi_n$  and  $\phi_{-n}$  are defined by

$$(3.6) \quad \Psi(z) = \sum_{n \geq 0} \psi_n z^{-n} = \exp\left(\sum_{n \in \mathbb{N}} \frac{(q^{2n} - q^{-2n})q^{n/2}}{n} a_n z^{-n}\right) q^{\partial}$$

$$(3.7) \quad \Phi(z) = \sum_{n \geq 0} \phi_{-n} z^n = \exp\left(\sum_{n \in \mathbb{N}} \frac{(q^{-2n} - q^{2n})q^{n/2}}{n} a_{-n} z^{-n}\right) q^{-\partial}$$

With the action of  $X_m^{\pm}$  the Chevalley generators are expressed by

$$\begin{aligned} e_1 &\rightarrow X_0^+, & f_1 &\rightarrow X_0^-, & K_1 &\rightarrow q^{\partial} \\ e_0 &\rightarrow X_1^- q^{-\partial}, & f_0 &\rightarrow q^{\partial} X_{-1}^+, & K_0 &\rightarrow q^{1-\partial} \end{aligned}$$

The vertex space is endowed with the standard inner product via

$$\begin{aligned} a_n^* &= a_{-n}, & (e^{\alpha})^* &= e^{-\alpha}, \\ (z^{\partial})^* &= z^{-\partial}. \end{aligned}$$

It follows from the commutation relations (3.1) that

$$(3.8) \quad (a_{-\lambda} e^{m\alpha} e^{i\alpha/2}, a_{-\mu} e^{n\alpha} e^{i\alpha/2}) = \delta_{mn} \delta_{\lambda\mu} z_{\lambda} \prod_{j \geq 1} \frac{1}{1 + q^{2\lambda_j}}$$

where  $z_\lambda$  as in section 2.

By Section 2 there are two special bases in  $V(\Lambda_i)$ : the power sum basis  $\{a_{-\lambda}e^{m\alpha}e^{i\alpha/2}\}$  and the Schur basis  $\{s_\lambda e^{m\alpha}e^{i\alpha/2}\}$ . However the Schur basis is no longer orthogonal with respect to the inner product (3.8).

Let  $b_{-n} = a_{-n}$ ,  $b_n = (1 + q^{2n})a_n$ ,  $n \in \mathbb{N}$ , then  $\{b_{-n}\}$  generate a standard Heisenberg algebra as in Section 2. In terms of the new Heisenberg generators we have

$$\begin{aligned} S(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} a_{-n} z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{1+q^{2n}}{n} a_n z^{-n}\right) = \sum_{n \in \mathbb{Z}} S_n z^{-n}, \\ S^*(z) &= \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} a_{-n} z^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{1+q^{2n}}{n} a_n z^{-n}\right) = \sum_{n \in \mathbb{Z}} S_n^* z^n, \end{aligned}$$

which generate the Schur function basis. For a partition  $\lambda$  and  $m \in \mathbb{Z}$  we define the *Schur symmetric polynomial* in  $V_i$  (cf. Theorem 2.4):

$$(3.9) \quad s_\lambda e^{m\alpha} e^{i\alpha/2} := S_{-\lambda} e^{m\alpha} e^{i\alpha/2} = S_{-\lambda_1} \cdots S_{-\lambda_l} e^{m\alpha} e^{i\alpha/2}.$$

The element  $s_\lambda$  is a polynomial over  $\mathbb{Q}$  in terms of the power sum  $a_\mu$ , where  $|\mu| = |\lambda|$ . Note that  $S(z)$  acts trivially on the lattice vector  $e^{m\alpha} e^{i\alpha/2}$ .

We need to recall some further terminology about partitions. Let  $\lambda$  and  $\mu$  be two partitions we write  $\lambda \supset \mu$  if the Young diagram of  $\lambda$  contains that of  $\mu$ . The set difference  $\lambda - \mu$  is called a skew diagram. The conjugate of a skew diagram  $\theta = \lambda - \mu$  is  $\theta' = \lambda' - \mu'$  and we define

$$(3.10) \quad |\theta| = \sum \theta_j = |\lambda| - |\mu|.$$

A skew diagram  $\theta$  is a horizontal  $n$ -strip (resp. a vertical  $n$ -strip) if  $|\theta| = n$  and  $\theta'_j \leq 1$  (resp.  $\theta_j \leq 1$ ) for each  $j$ . Thus a horizontal (resp. vertical) strip has at most one column (resp. rows) in its diagram. We will also denote by  $(\lambda, \mu)$  the juxtaposition of  $\lambda$  and  $\mu$ . As before  $\pi(\lambda, \mu)$  is the partition associated to  $(\lambda, \mu)$  by performing the linkage symmetry.

For a partition  $\mu$  and an integer  $m$  we let  $\mathcal{V}_n = \mathcal{V}_n(m, \mu)$  be the set of the partitions of  $\lambda$  such that the skew diagram  $\lambda - (m, \mu)'$  is a vertical  $n$ -strip. We also let  $\mathcal{H}_n = \mathcal{H}_n(m, \mu)$  be the set of partitions  $\lambda$  such that the skew diagram  $\lambda - (m, \mu)$  is a horizontal  $n$ -strip. Note that  $\mathcal{V}_n$  may be described as the set of partitions  $\lambda$  such that the skew diagram  $\lambda - (1^m, \mu)$  is a vertical  $n$ -strip. The following is called the Pieri rule [M]:

$$(3.11) \quad s_n S_{-m} S_{-\mu} \cdot 1 = \sum_{\lambda \in \mathcal{H}_n(m, \mu)} \text{sgn}(m, \mu) s_\lambda,$$

$$(3.12) \quad s_{1^n} S_{-m} S_{-\mu} \cdot 1 = \sum_{\lambda \in \mathcal{V}_n(m, \mu)} (-1)^n \text{sgn}(m, \mu')' s_\lambda.$$

We will also frequently use  $\lambda - \mu$  to denote the difference of two integral vectors in  $\mathbb{Z}^n$ .

**Theorem 3.2.** *The quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  is realized on the Fock space  $\Lambda \otimes \mathbb{C}[\mathbb{Z}\alpha]e^{i\alpha/2}$  of symmetric functions by the following action:*

$$X_n^+ s_\mu e^{m\alpha} e^{i\alpha/2} = \left( \sum_{j=0}^{l(\mu)-2m-n-1-i} q^{-2m-n-1-i-2j} \cdot \right. \\ \left. \operatorname{sgn}(-2m-n-1-i-j, \mu) \sum_{\lambda \in \mathcal{H}_j} s_\lambda \right) e^{(m+1)\alpha} e^{i\alpha/2}.$$

where  $\mathcal{H}_j = \mathcal{H}_j(-2m-n-1-i-j, \mu)$ , the sign refers to  $\operatorname{sgn}(-2m-n-1-i-j, \mu) = (-1)^{l(\sigma)}$  such that  $(-2m-n-1-i-j, \mu) + \delta = \sigma(\lambda + \delta)$  and  $\lambda$  is the partition of length at most  $l(\mu) + 1$ . Also we have

$$X_n^- s_\mu e^{m\alpha} e^{i\alpha/2} = (-1)^{n+1+i} \left( \sum_{j=0}^{\mu_1+2m-n-1+i} q^{2j} \cdot \right. \\ \left. \operatorname{sgn}(2m-n-1-j+i, \mu') \sum_{\lambda \in \mathcal{V}_j} s_\lambda \right) e^{(m-1)\alpha} e^{i\alpha/2}$$

where  $\mathcal{V}_j = \mathcal{V}_j(2m-n-1-j+i, \mu')$ , the sign refers to  $\operatorname{sgn}(2m-n-1-j+i, \mu) = (-1)^{l(\sigma)}$  such that  $(2m-n-1-j+i, \mu) + \delta = \sigma(\lambda + \delta)$  and  $\lambda$  is the partition of length at most  $l(\mu) + 2m-n-1-j+i$ .

*Proof.* This follows from our vertex operator calculus of symmetric functions [J1, J2].

$$\begin{aligned} & X^+(z) S(w_1) \cdots S(w_l) e^{m\alpha} e^{i\alpha/2} \\ & =: X^+(z) S(w_1) \cdots S(w_l) : z^{2m+i} e^{(m+1)\alpha} e^{i\alpha/2} \prod_j (1 - q^{-1} \frac{w_j}{z}) \prod_{j < k} (1 - \frac{w_k}{w_j}) \\ & = \exp\left(\sum_{n=1}^{\infty} \frac{q^{-2n}}{n} a_{-n} w_0^n\right) S(w_0) \cdots S(w_l) e^{(m+1)\alpha} e^{i\alpha/2} w_0^{2m+i} q^{-2m-i}, \end{aligned}$$

where  $w_0 = qz$ . Taking the coefficient of  $z^{-n-1} w^{-\mu}$  and using Littlewood-Richardson rule (2.2) we obtain the result. The case of  $X_n^- s_\mu e^{m\alpha}$  is proved similarly.

$$\begin{aligned} & X^-(z) S(w_1) \cdots S(w_l) e^{m\alpha} e^{i\alpha/2} \\ & =: X^-(z) S(w_1) \cdots S(w_l) : z^{-2m-i} e^{(m-1)\alpha} e^{i\alpha/2} \prod_{j < k} (1 - \frac{w_k}{w_j}) \prod_j (1 - \frac{w_j}{z})^{-1} \\ & = S^*(q^2 z) \cdot 1 S^*(z) S(w_1) \cdots S(w_l) z^{-2m-i} e^{(m-1)\alpha} e^{i\alpha/2}. \end{aligned}$$

Taking the coefficient of  $z^{-n-1} w^\mu$  we obtain the formula.  $\square$

We can reformulate the result in terms of the standard inner product in Section 2. Let  $u_j = (0, \dots, 0, 1, 0, \dots, 0)$  be the  $j$ th unit vector in  $\mathbb{Z}^n$ . Let  $\mathbf{1}_{(l_1, \dots, l_j)}$  be the sum of the unit vectors  $u_{l_1}, \dots, u_{l_j}$ .

**Proposition 3.1.** *For  $n \in \mathbb{Z}$  and a partition  $\mu$  we have*

$$\begin{aligned} & X_n^- s_\mu e^{m\alpha} e^{i\alpha/2} \\ &= \sum_{\lambda} s_{\lambda} e^{(m-1)\alpha} e^{i\alpha/2} \sum_{j=0}^{l(\lambda)} (-q^2)^j \sum_{l_1 < \dots < l_j} (S_{2m+i-n-1-j}^* S_{-\mu}, S_{-(\lambda - \mathbf{1}_{(l_1, \dots, l_j)})}), \end{aligned}$$

where  $\lambda$  runs through partitions of weight  $|\mu| + 2m - n - 1 + i$  such that  $\lambda - \mathbf{1}_{(l_1, \dots, l_j)}$  is the juxtaposition of  $(1^{2m-n-j-1+i})$  and  $\mu$ .

Later in Theorem 3.5 we will give another proof in terms of the dual vertex operator  $S^*(z)$ .

**Example 3.3.** Using Theorem 3.2 it is easy to compute the following.

$$\begin{aligned} X_n^{\pm} e^{r\alpha} e^{i\alpha/2} &= 0, \quad \text{if } n > \mp 2r - 1 \mp i, \\ X_{-2r+1-i}^+ \cdots X_{-3-i}^+ X_{-1-i}^+ e^{i\alpha/2} &= e^{r\alpha} e^{i\alpha/2}, \quad r \geq 1, \\ X_{2r-3+i}^- \cdots X_{1+i}^+ X_{-1+i}^+ e^{i\alpha/2} &= e^{-r\alpha} e^{i\alpha/2}, \quad r \geq 1. \end{aligned}$$

$$\begin{aligned} X_{-1}^+ s_{(2,1)} e^{-\alpha} &= q^2 s_{(2,2,1)} - q^{-2} (s_5 + s_{(4,1)} + s_{(3,2)}) + q^{-6} (s_5 + s_{(4,1)}), \\ X_0^- s_1 e^{\alpha} &= -s_{(2)} + q^4 s_{(1^2)}. \end{aligned}$$

We can generalize the action to the divided powers of  $X_n^{\pm(r)}$ .

**Lemma 3.4.** [BFJ] *For  $r \in \mathbb{N}$  we have*

$$\prod_{1 \leq i < j \leq k} (z_i - qz_j) = \sum_{w \in \mathfrak{S}_k} (-q)^{\ell(w)} z^{w(\delta)} + \sum a_{\gamma_1, \dots, \gamma_k} z_1^{\gamma_1} z_2^{\gamma_2} \cdots z_k^{\gamma_k},$$

where  $\delta = (k-1, k-2, \dots, 0)$  and the second sum consists of certain monomials such that some  $\gamma_i = \gamma_j$ ,  $i \neq j$  and  $a_{\gamma} \in \mathbb{Z}[q]$ ,  $a_{\gamma}(1) = 0$ .

**Theorem 3.5.** *The lattices  $\mathcal{L}_i = \sum_{\lambda, m} \oplus \mathbb{Z}[q, q^{-1}] s_{\lambda} e^{m\alpha} e^{i\alpha/2}$  are invariant under the action of the divided powers  $X_n^{\pm(r)}$ . More precisely we have*

$$\begin{aligned} & X_n^{+(r)} s_{\mu} e^{m\alpha} e^{i\alpha/2} \\ &= q^{-3\binom{r}{2} - r(n+1+2m+i)} \sum_{l(\lambda) \leq r} q^{-2|\lambda|} s_{\lambda} \cdot s_{(-\lambda - 2\delta - (n+1+2m+i)\mathbf{1}, \mu)} e^{(m+r)\alpha} e^{i\alpha/2} \\ & X_n^{-(r)} s_{\mu'} e^{m\alpha} e^{i\alpha/2} \\ &= (-1)^{r(n+1+i)} q^{\binom{r}{2}} \sum_{l(\lambda) \leq r} q^{2|\lambda|} s_{\lambda'} \cdot s_{(-\lambda - 2\delta - (n+1-2m-i)\mathbf{1}, \mu)'} e^{(m-r)\alpha} e^{i\alpha/2} \end{aligned}$$

where the summations run through all partitions  $\lambda$  of length  $\leq r$ ,  $\delta = (r-1, r-2, \dots, 0)$ , and  $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^r$ .

*Proof.* Let  $z = (z_1, \dots, z_r)$ ,  $w = (w_1, \dots, w_l)$ , and  $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^r$  in the following computation.

$$\begin{aligned}
& x_n^{+r} s_\mu e^{m\alpha} e^{i\alpha/2} \\
&= \oint X^+(z_1) \cdots X^+(z_r) S(w_1) \cdots S(w_l) z^{(n+1)\mathbf{1}} w^{-\mu} \frac{dz dw}{zw} e^{m\alpha} e^{i\alpha/2} \\
&= \oint \exp\left(\sum_{n=1}^{\infty} \frac{(q^n + q^{-n})}{n} a_{-n}(z_1^n + \cdots + z_r^n)\right) : S(w_1) \cdots S(w_l) : \\
&\quad \times \prod_{i < j} (z_i - z_j)(z_i - q^{-2}z_j) \left(1 - \frac{w_j}{w_i}\right) \prod_{i,j} \left(1 - q^{-1} \frac{w_j}{z_i}\right) \\
&\quad \times z^{(2m+n+1+i)\mathbf{1}} e^{(m+r)\alpha} e^{i\alpha/2} \frac{dz}{z} \frac{dw}{w}.
\end{aligned}$$

Note that the integrand divided by  $\prod_{j < k} (z_j - q^{-2}z_k)$  is an anti-symmetric function in  $z_1, \dots, z_r$ . It follows from Lemma 3.4 that the terms  $z_1^{\gamma_1} z_2^{\gamma_2} \cdots z_r^{\gamma_r}$  (for which some  $\gamma_k = \gamma_j$ ) make no contribution to the integral. Therefore

$$\begin{aligned}
x_n^{+(r)} s_\mu e^{m\alpha} e^{i\alpha/2} &= \frac{1}{[r]!} \sum_{w \in \mathfrak{S}_r} \oint \exp\left(\sum_{n=1}^{\infty} \frac{(q^n + q^{-n})}{n} a_{-n}(z_1^n + \cdots + z_r^n)\right) \\
&\quad \times : S(w_1) \cdots S(w_l) : \prod_{i < j} (z_i - z_j) \prod_{i < j} \left(1 - \frac{w_j}{w_i}\right) \prod_{i,j} \left(1 - q^{-1} \frac{w_j}{z_i}\right) \\
&\quad (-q)^{-\ell(w)} z^{w(\delta) + (2m+n+1+i)\mathbf{1}} w^{-\mu} e^{(m+r)\alpha} e^{i\alpha/2} \frac{dz dw}{zw} \\
&= q^{-\binom{r}{2}} \oint \exp\left(\sum_{n=1}^{\infty} \frac{q^{-2n}}{n} a_{-n}(z_1^n + \cdots + z_r^n)\right) S(qz_1) \cdots S(qz_r) \\
&\quad S(w_1) \cdots S(w_l) z^{2\delta + (2m+n+1+i)\mathbf{1}} w^{-\mu} e^{(m+r)\alpha} e^{i\alpha/2} \frac{dz dw}{zw},
\end{aligned}$$

where  $\delta = (r-1, r-2, \dots, 0)$  and we have used

$$\sum_{w \in \mathfrak{S}_r} q^{-2\ell(w)} = q^{-\binom{r}{2}} [r]!.$$

From the orthogonality of Schur functions [M] it follows that

$$\exp\left(\sum_{n=1}^{\infty} \frac{a_{-n}}{n} (z_1^n + \cdots + z_r^n)\right) = \sum_{l(\lambda) \leq r} s_\lambda(a_{-k}) s_\lambda(z_i)$$

where  $s_\lambda(a_{-k})$  is the Schur function in terms of the power sum  $a_{-\mu}$  and  $s_\lambda(z_i)$  is the Schur polynomial in the variables  $z_1, \dots, z_r$ . Replacing  $z_j$  by  $q^{-1}z_j$  we get

$$\begin{aligned}
& x_n^{+(r)} s_\mu e^{m\alpha} e^{i\alpha/2} \\
&= q^{-3\binom{r}{2}-r(n+1+2m+i)} \sum_{l(\lambda) \leq r} q^{-2|\lambda|} s_\lambda \oint s_\lambda(z) z^{2\delta+(n+1+2m+i)\mathbf{1}} \\
&\quad \times S(z_1) \cdots S(z_r) S(w_1) \cdots S(w_l) e^{(m+r)\alpha} e^{i\alpha/2} w^{-\mu} \frac{dz}{z} \frac{dw}{w} \\
&= q^{-3\binom{r}{2}-r(n+1+2m+i)} \sum_{l(\lambda) \leq r} s_\lambda \oint \sum_{w \in \mathfrak{S}_r} \frac{(-1)^{l(w)} z^{w(\lambda+\delta)}}{\prod_{j < k} (z_j - z_k)} z^{\delta+(n+1+2m+i)\mathbf{1}} \\
&\quad \times z^\delta S(z_1) \cdots S(z_r) S(w_1) \cdots S(w_l) e^{(m+r)\alpha} e^{i\alpha/2} w^{-\mu} \frac{dz}{z} \frac{dw}{w}.
\end{aligned}$$

Applying the symmetry of the Schur vertex operators in Proposition 2.1 we see that the above expression becomes

$$\begin{aligned}
& q^{-3\binom{r}{2}-r(n+1+2m+i)} \sum_{l(\lambda) \leq r} q^{-2|\lambda|} s_\lambda \sum_{w \in \mathfrak{S}_r} \oint \frac{(-1)^{l(w)} z^{\lambda+\delta}}{\prod_{j < k} (z_j - z_k)} z^{w(\delta)+(n+1+2m+i)\mathbf{1}} \\
&\quad \times z^\delta S(z_1) \cdots S(z_r) S(w_1) \cdots S(w_l) e^{(m+r)\alpha} e^{i\alpha/2} w^{-\mu} \frac{dz}{z} \frac{dw}{w} \\
&= q^{-3\binom{r}{2}-r(n+1+2m+i)} \sum_{l(\lambda) \leq r} q^{-2|\lambda|} s_\lambda \oint z^{\lambda+2\delta+(n+1+2m+i)\mathbf{1}} \\
&\quad \times S(z_1) \cdots S(z_r) S(w_1) \cdots S(w_l) e^{(m+r)\alpha} e^{i\alpha/2} w^{-\mu} \frac{dz}{z} \frac{dw}{w},
\end{aligned}$$

where we have used the Weyl denominator formula (see  $\lambda = 0$  in (2.1)) and the integral is taken along contours in  $z_i, w_i$  around the origin. The formula for  $X_n^{+(r)}$  is then obtained by using Theorem 2.4.

The case of  $X_n^{-(r)}$  is proved similarly with the help of the dual vertex operator  $S^*(z)$ .

$$\begin{aligned}
& X_n^{-(r)} s_{\mu'} e^{m\alpha} e^{i\alpha/2} \\
&= \frac{(-1)^{|\mu|}}{[r]!} \oint X^-(z_1) \cdots X^-(z_r) S^*(w_1) \cdots S^*(w_l) z^{n+1} w^{-\mu} e^{(m-r)\alpha} e^{i\alpha/2} \frac{dz dw}{zw} \\
&= \frac{(-1)^{|\mu|}}{[r]!} \oint : S^*(q^2 z_1) \cdots S^*(q^2 z_r) : S^*(z_1) \cdots S^*(z_r) S^*(w_1) \cdots S^*(w_l) \\
&\quad \times \prod_{i < j} (z_i - q^2 z_j) z^{\delta+(n+1-2m-i)\mathbf{1}} w^{-\mu} e^{(m-r)\alpha} e^{i\alpha/2} \frac{dz dw}{zw} \\
&= q^{\binom{r}{2}} \oint : S^*(q^2 z_1) \cdots S^*(q^2 z_r) : S^*(z_1) \cdots S^*(z_r) S^*(w_1) \cdots S^*(w_l) \\
&\quad \times z^{2\delta+(n+1-2m-i)\mathbf{1}} w^{-\mu} e^{(m-r)\alpha} e^{i\alpha/2} \frac{dz dw}{zw} (-1)^{|\nu|}
\end{aligned}$$

where we have used the skew-symmetry of the integrand and Lemma 3.4. Then the formula for  $X_n^{-(r)}$  is obtained from the following identity and the Weyl denominator

formula.

$$\exp\left(-\sum_{n=1}^{\infty} \frac{a_{-n}}{n} (z_1^n + \cdots + z_r^n)\right) = \sum_{l(\lambda) \leq r} s_{\lambda'}(a_{-k}) s_{\lambda}(z_i)$$

□

#### 4. COMBINATORIAL REALIZATION OF $U_q(\widehat{sl}_2)$

Let  $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ . We denote by  $U_q(\widehat{sl}_2)_{\mathcal{A}}$  the  $\mathcal{A}$ -algebra generated by  $e_i^{(n)}, f_i^{(n)}, K_i^{\pm 1}$ . From section 3 it follows that

$$(4.1) \quad \begin{aligned} e_1^{(r)} s_{\mu} e^{m\alpha} e^{i\alpha/2} &= q^{-3\binom{r}{2} - r(2m+i+1)} \\ &\times \left( \sum_{l(\lambda) \leq r} q^{-2|\lambda|} s_{\lambda} s_{(-\lambda-2\delta-(2m+1+i)\mathbf{1}, \mu)} \right) e^{(m+r)\alpha} e^{i\alpha/2} \end{aligned}$$

$$(4.2) \quad \begin{aligned} f_1^{(r)} s_{\mu'} e^{m\alpha} e^{i\alpha/2} &= (-1)^{r(1+i)} q^{\binom{r}{2}} \\ &\times \left( \sum_{l(\lambda) \leq r} q^{2|\lambda|} s_{\lambda'} s_{(-\lambda-2\delta+(2m+i-1)\mathbf{1}, \mu)'} \right) e^{(m-r)\alpha} e^{i\alpha/2} \end{aligned}$$

$$(4.3) \quad \begin{aligned} f_0^{(r)} s_{\mu} e^{m\alpha} e^{i\alpha/2} &= q^{r(5-r)/2} \\ &\times \left( \sum_{l(\lambda) \leq r} q^{-2|\lambda|} s_{\lambda} s_{(-\lambda-2\delta-(2m+i)\mathbf{1}, \mu)} \right) e^{(m+r)\alpha} e^{i\alpha/2} \end{aligned}$$

$$(4.4) \quad \begin{aligned} e_0^{(r)} s_{\mu'} e^{m\alpha} e^{i\alpha/2} &= (-1)^{ri} q^{-\binom{r}{2} - r(2m+i)} \\ &\times \left( \sum_{l(\lambda) \leq r} q^{2|\lambda|} s_{\lambda'} s_{(-\lambda-2\delta+(2m+i-2)\mathbf{1}, \mu)'} \right) e^{(m-r)\alpha} e^{i\alpha/2} \end{aligned}$$

As a consequence of these formulas and the Littlewood-Richardson rule (2.2) we get the following theorem.

**Theorem 4.1.** *The  $\mathcal{A}$ -lattice generated by the Schur functions  $\{s_{\mu} e^{m\alpha}\}$ ,  $m \in \mathbb{Z}, \mu \in \mathcal{P}$ , is invariant under the integral form  $U_q(\widehat{sl}_2)_{\mathcal{A}}$ .*

**Proposition 4.1.** *For  $m \geq 0$  we have*

$$\begin{aligned} f_1^{(2m)} e^{m\alpha} &= (-1)^m q^{m(2m-1)} e^{-m\alpha}, \\ f_0^{(2m+1)} e^{-m\alpha} &= (-1)^m q^{-(2m+1)(m-2)} e^{-(m+1)\alpha}, \\ f_0^{(2m)} e^{-m\alpha} e^{\alpha/2} &= (-1)^m q^{-m(2m-5)} e^{m\alpha}, \\ f_1^{(2m+1)} e^{m\alpha} e^{\alpha/2} &= (-1)^m q^{m(2m+1)} e^{-(m+1)\alpha}. \end{aligned}$$

*Proof.* The four formulas are proved similarly. Take  $f_1^{(2m)} e^{m\alpha}$ . Observe that  $-2\delta + (2m-1)\mathbf{1} = (-2m+1, -2m+3, \dots, 2m-3, 2m-1)$  is of weight zero, thus only  $\lambda = 0$  contributes to the summation in  $f_1^{(2m)} e^{m\alpha}$  (see (4.2)). Since the longest element in  $\mathfrak{S}_{2m}$  has inversion number  $m(m-1)$ , the sign of  $s_{-2\delta+(2m-1)\mathbf{1}}$  is  $(-1)^{m(m-1)} = (-1)^m$ . □

**Corollary 4.1.** *For  $m \geq 0$  we have*

$$\begin{aligned} f_1^{(2m)} f_0^{(2m-1)} \dots f_1^{(2)} f_0.1 &= (-1)^m q^{3m^2} e^{-m\alpha}, \\ f_0^{(2m+1)} f_1^{(2m)} \dots f_1^{(2)} f_0.1 &= q^{(m+1)(m+2)} e^{(m+1)\alpha}, \\ f_0^{(2m)} f_1^{(2m-1)} \dots f_0^{(2)} f_1 e^{\alpha/2} &= (-1)^m q^{m(m+2)} e^{m\alpha} e^{\alpha/2}, \\ f_1^{(2m+1)} f_0^{(2m)} \dots f_0^{(2)} f_1 e^{\alpha/2} &= q^{3m(m+1)} e^{-(m+1)\alpha} e^{\alpha/2}, \end{aligned}$$

**Example 4.2.** In the following we abbreviate  $f_{i_1}^{(n_1)} \dots f_{i_r}^{(n_r)}.1 = f_{i_1}^{(n_1)} \dots f_{i_r}^{(n_r)}$  in the basic representation  $V(\Lambda_0)$ .

$$\begin{aligned} f_0 &= q^2 e^\alpha \\ f_1 f_0 &= -q^2 (1 + q^2) s_1 \\ f_1^{(2)} f_0 &= -q^3 e^{-\alpha} \\ f_0 f_1 f_0 &= q^2 (q^2 + 1) s_1 e^\alpha \\ f_1 f_0 f_1 f_0 &= -(q^4 + q^2) (s_2 - q^4 s_{1^2}) \\ f_0 f_1^{(2)} f_0 &= -q^3 (s_{1^2} + [3] s_2) \\ f_0 f_1 f_0 f_1 f_0 &= q^2 (1 + q^2)^2 (s_2 + s_{1^2}) e^\alpha \\ f_0^{(2)} f_1^{(2)} f_0 &= q^4 (s_2 + [3] s_{1^2}) e^\alpha \\ f_0^{(3)} f_1^{(2)} f_0 &= q^6 e^{2\alpha} \\ f_1^{(2)} f_0 f_1 f_0 &= q^5 (1 + q^2) s_1 e^{-\alpha} \end{aligned}$$

**Example 4.3.** As in the last example we use  $f_{i_1}^{(n_1)} \dots f_{i_r}^{(n_r)}$  to denote  $f_{i_1}^{(n_1)} \dots f_{i_r}^{(n_r)} e^{\alpha/2}$  in the basic representation  $V(\Lambda_1)$ .

$$\begin{aligned} f_1 &= e^{-\alpha} \\ f_0 f_1 &= q^{-2} (1 + q^{-2}) s_1 \\ f_0^{(2)} f_1 &= -q^3 e^\alpha \\ f_1 f_0 f_1 &= -(q^{-2} + 1) s_1 e^{-\alpha} \\ f_0 f_1 f_0 f_1 &= (1 + q^2) (s_2 - s_{1^2}) \\ f_1 f_0^{(2)} f_1 &= -q^5 ([3] s_{1^2} + s_2) \\ f_1 f_0 f_1 f_0 f_1 &= (1 + q^2)^2 (s_2 + q^2 s_{1^2}) e^{-\alpha} \\ f_1^{(2)} f_0^{(2)} f_1 &= q^5 ([3] s_2 + s_{1^2}) e^{-\alpha} \\ f_1^{(3)} f_0^{(2)} f_1 &= q^6 e^{-2\alpha} \\ f_0^{(2)} f_1 f_0 f_1 &= -[2] s_1 e^\alpha \end{aligned}$$

It would be interesting to see the relation between our formulas and the fermionic picture [LLT].



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